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On Reeb components of invariant foliations of projectively Anosov flows ☆

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Abstract

We show that if a C^2 codimension one foliation on a three-dimensional manifold has a Reeb component and is invariant under a projectively Anosov flow, then it must be a Reeb foliation on $S^2 \times S^1$.

MSC: primary 37D30; secondary 57R30

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1. Introduction

In [3], Eliashberg and Thurston developed a theory of confoliations, which are mixture of foliations and contact structures on three-dimensional manifolds. They introduced a special class of deformations of foliations into contact structures, which is so-called *linear perturbations*. Suppose that a codimension one foliation $\mathcal{F}$ is generated by the kernel of a 1-form α. We say that a family $\{\alpha_t\}_{t\in\mathbb{R}}$ of one forms is *a linear deformation into a contact structures* of $\mathcal{F}$ if the kernel of α_0 generates $\mathcal{F}$ and $(d/dt)(\alpha_t \wedge d\alpha_t)|_{t=0}$ is a positive volume form. It is easy to see that α_t is a positive (respectively negative) contact form for any $t > 0$ (respectively $t < 0$) sufficiently close to 0. Eliashberg and Thurston observed that if the kernel of $d\alpha_t/dt|_{t=0}$ also generates a foliation, then the kernels of α_t and α_{-t} are mutually transverse and their intersection generates a flow with a special property. In [5], Mitsumatsu studied that the same deformation for invariant foliations of Anosov flows. He called a pair of mutually transverse positive and negative contact structures *a bi-contact structure*, and a flow corresponding to a bi-contact structure *a projectively Anosov* flow or simply $\mathbb{P}$A flow (Eliashberg and Thurston called it a *conformally Anosov* flow).

Let M be a closed three-dimensional manifold and $\Phi = \{\Phi^t\}_{t\in\mathbb{R}}$ a flow on M without stationary points. A decomposition $TM = E^u + E^s$ by continuous two-dimensional subbundles of TM is called *a projectively Anosov* (*or simply* $\mathbb{P}$A) *splitting* if

(1) $E^u \cap E^s = T\Phi$, where $T\Phi$ is the one-dimensional subbundle of TM that is tangent to the orbits of Φ,

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(2) both E^u and E^s are $D\Phi$-invariant, and
(3) there exist two constants $C > 0$ and $\lambda \in (0, 1)$ such that

$$\left\| D\hat{\Phi}^{-t}|_{(E^u/T\Phi)(\Phi^t(z))} \right\| \cdot \left\| D\hat{\Phi}^t|_{(E^s/T\Phi)(z)} \right\| \leqslant C\lambda^t$$

for any $t > 0$ and $z \in M$, where $D\hat{\Phi}$ is the flow on $TM/T\Phi$ induced from Φ.

It is easy to see that the definition does not depend on the choice of the norm $\|\cdot\|$ and that the decomposition $TM = E^u + E^s$ is uniquely determined if it exists. We say a flow is *projectively Anosov* (*or* $\mathbb{P}$A) when it admits a $\mathbb{P}$A splitting. Mitsumatsu [5] showed that a flow Φ without stationary points is a $\mathbb{P}$A flow if and only if there exists a bi-contact structure (ξ, η) satisfying $T\Phi = \xi \cap \eta$. It is known that E^u and E^s are integrable but not uniquely integrable in general. If E^s or E^u generates a C^r foliation, then it is *invariant under the flow*, that is, every leaf is an invariant set of the flow. It is known that any foliation invariant under a $\mathbb{P}$A flow must be tangent to E^s or E^u.

In this paper, we consider the conditions that characterize invariant foliations of $\mathbb{P}$A flows. It is important from the view point of linear perturbation of foliations. In fact, a foliation is invariant under a $\mathbb{P}$A flow if and only if it admits a linear perturbation $\{\alpha_t\}$ into contact structure which changes its direction with non-zero speed everywhere, that is, the kernel of $d\alpha_t/dt|_{t=0}$ is transverse to the foliation. Eliashberg and Thurston showed that if a foliation has holonomy and each compact leaf has non-trivial linear holonomy, then it admits a linear perturbation into contact structures. So, it is natural to ask which foliation admits a linear perturbation into contact structures changing its direction with non-zero speed everywhere.

We say a $\mathbb{P}$A flow is *regular*, if both E^u and E^s generate smooth invariant foliations. In [8–10,12], Noda and Tsuboi gave a classification of regular $\mathbb{P}$A flows for some special three-dimensional manifolds and the author [2] finally completed the classification for any three-dimensional manifolds. The classification implies that any invariant foliation of a regular $\mathbb{P}$A flow is either an invariant foliation of an algebraic Anosov flow or decomposed into the sum of the foliations $\mathcal{F}([\omega], 1)$ and $\mathcal{F}([\omega], 2)$ on $\mathbb{T}^2 \times [0, 1]$ that are given by Moussu and Roussarie [7].

Mitsumatsu [6] showed that the invariant foliations of any C^1-regular $\mathbb{P}$A flow have no Reeb component. On the other hand, Minakawa [4] constructed a $\mathbb{P}$A flow which is tangent to a Reeb foliation on $S^2 \times S^1$. The main theorem of this paper asserts that any C^2 foliation invariant under a $\mathbb{P}$A flow have no Reeb component with the only one exception.

Theorem 1.1. *Let Φ be a $\mathbb{P}$A flow on a closed and connected three-dimensional manifold M. If a C^2 foliation $\mathcal{F}$ with a Reeb component is invariant under a $\mathbb{P}$A flow, then $\mathcal{F}$ is a Reeb foliation on $S^2 \times S^1$ with non-trivial linear holonomy.*

Remark that it is still unknown whether the main theorem holds for C^1 foliations or not.

Let $\mathbb{T}^n$ denote the n-dimensional torus. We can apply Theorem 1.1 to the classification of invariant foliations of $\mathbb{P}$A flows on $\mathbb{T}^3$.

Corollary 1.2. *If a C^2 foliation on $\mathbb{T}^3$ is transversely orientable and is invariant under a $\mathbb{P}$A flow, then it is decomposed into the sum of the foliations $\mathcal{F}([\omega], 1)$ and $\mathcal{F}([\omega], 2)$ on $\mathbb{T}^2 \times [0, 1]$.*

In fact, the result of Eliashberg and Thurston explained above implies that such a foliation has holonomy and any compact leaf has non-trivial linear holonomy. Hence, the corollary follows from the classification of C^2 foliations on $\mathbb{T}^3$ without Reeb components by Moussu and Roussarie [7].

We pose a conjecture to end the introduction.

Conjecture. *Suppose that a smooth foliation on a closed, connected, and three-dimensional manifold is invariant under a $\mathbb{P}$A flow, then it is equivalent to one of the followings up to finite covering*:

(1) *The Reeb foliation on $S^2 \times S^1$.*
(2) *An invariant foliation of an algebraic Anosov flow.*
(3) *A foliation decomposed into the sum of the foliations $\mathcal{F}([\omega], 1)$ and $\mathcal{F}([\omega], 2)$.*

2. Proof of Theorem 1.1

We fix a $\mathbb{P}$A flow Φ on a closed, connected, and three-dimensional manifold M. Let $TM = E^u + E^s$ be the $\mathbb{P}$A splitting for Φ. Suppose that a C^2-foliation $\mathcal{F}^s$ is invariant under Φ. Without loss of generality, we may assume that $\mathcal{F}^s$ is tangent to E^s. It is known that the set of $\mathbb{P}$A flows is an open subset of the set of all C^1 flows on M. Hence, replacing Φ with its approximation which preserves $\mathcal{F}^s$, we may assume that Φ is of class C^2. Without loss of generality, we also assume that $\mathcal{F}^s$ is transversely orientable.

Let $\mathcal{O}(z)$ denote the orbit $\{\Phi^t(z) \mid t \in \mathbb{R}\}$ of $z \in M$. For a compact Φ-invariant subset Λ of M, we define *the stable set* $W^s(\Lambda)$ and *the unstable set* $W^u(\Lambda)$ by

$$W^s(\Lambda) = \Big\{z \in M \mid \lim_{t\to+\infty} d(\Phi^t(z), \Lambda) = 0\Big\},$$
$$W^u(\Lambda) = \Big\{z \in M \mid \lim_{t\to-\infty} d(\Phi^t(z), \Lambda) = 0\Big\}.$$

For a periodic point z of Φ and $\tau \in \{u, s\}$, we put $\lambda^\tau(z) = \|D\hat{\Phi}^{t_z}|_{(E^\tau/T\Phi)(z)}\|$, where t_z is the period of z. Notice that $\lambda^u(z) > \lambda^s(z)$ for any periodic point z. We say z is *an attracting periodic point* if $\lambda^u(z) < 1$. It is known that $W^s(\mathcal{O}(z))$ is an open neighborhood of $\mathcal{O}(z)$ for any attracting periodic point z.

We say a closed leaf L of $\mathcal{F}^s$ is *normally repelling* when there exists $C > 0$ and $\lambda \in (0, 1)$ such that $\|D\hat{\Phi}^{-t}|_{(E^u/T\Phi)(z)}\| < C\lambda^t$ for any $z \in L$ and any $t > 0$. Let $\Omega^s_*(\Phi)$ be the union of closed leaves of $\mathcal{F}^s$ on which the restriction of Φ is topologically conjugate to a linear flow on the two-dimensional torus. It is known that for a normally repelling leaf L, the unstable set $W^u(L)$ is an open neighborhood of L and that $\Omega^s_*(\Phi)$ consists of finitely many normally repelling leaves.

2.1. Topology of the stable set

We call a C^2-embedding $\psi : [-2, 2]^2 \times \Sigma \to M$ with a finite set Σ a *canonical cross section* if

(1) $\operatorname{Im} D\psi_w \cap T\Phi(w) = \{0\}$ and $D\psi_w(\partial/\partial x) \in T_{\psi(w)}\mathcal{F}^s$ for any $w \in [-2, 2]^2$, and
(2) both $\{\Phi^t(z) \mid t > 0\}$ and $\{\Phi^t(z) \mid t < 0\}$ intersect with $\psi([-1, 1]^2 \times \Sigma)$ for any $z \in M$.

It is easy to see that a canonical cross section exists. In the rest of the proof, we fix a canonical cross section ψ. We say a subset R of $[-2, 2] \times \Sigma$ is a *rectangle* if it has the form $[x_-, x_+] \times [y_-, y_+] \times \sigma_0$.

We call a C^2-diffeomorphism $r : R \to R'$ between two rectangles R and R' a *return* of (Φ, ψ) if there exists a positive valued function τ on R such that $\Phi^{\tau(w)}(\psi(w)) = \psi \circ r(w)$ for any $w \in R$. Remark that $r^{-1} : R' \to R$ is a return of $(\Phi^{-1} = \{\Phi^{-t}\}_{t\in\mathbb{R}}, \psi)$.

Let $\mathcal{R} = \{r_k : R_k \to R'_k\}_{k=1}^{k_*}$ be a family of returns of (Φ, ψ). We say a sequence $(k(n))_{n=1}^{n_*}$ with $n_* \in \{1, 2, \ldots, \infty\}$ is *$\mathcal{R}$-admissible* for a subset Λ of $[-2, 2]^2 \times \Sigma$ when $r_{k(n)} \circ \cdots \circ r_{k(1)}|_\Lambda$ is well-defined for any $n = 1, 2, \ldots, n_*$. An $\mathcal{R}$-admissible sequence $(k(n))_{n=1}^{n_*}$ is said to be *fine* if $\Lambda \subset [-1, 1]^2 \times \Sigma$ and $r_{k(n)} \circ \cdots \circ r_{k(1)}(\Lambda) \subset [-1, 1]^2 \times \Sigma$ for any $n = 1, 2, \ldots, n_*$.

We write $\mathcal{R}^{-1}$ for a family of returns $\{r_k^{-1}\}_{k=1}^{k_*}$ associated to (Φ^{-1}, ψ). We say the family $\mathcal{R}$ is *full* when

(1) $[-1, 1] \times \Sigma \subset (\bigcup_{k=1}^{k_*} R_k) \cap (\bigcup_{k=1}^{k_*} R'_k)$, and
(2) there exists $\Delta_0 > 0$ such that if $w \in R_k \cap [-1, 1]^2 \times \Sigma$ and $r_k(w) \in R'_k \cap [-1, 1]^2 \times \Sigma$ then $Q_{\Delta_0}(w) \subset R_k$ and $Q_{\Delta_0}(r_k(w)) \subset R'_k$, where $Q_\Delta(x, y, \sigma) = [x - \Delta, x + \Delta] \times [y - \Delta, y + \Delta] \times \sigma$.

It is easy to see that there exist a full family of returns of (Φ, ψ). We fix such a family $\mathcal{R} = \{r_k : R_k \to R'_k\}_{k=1}^{k_*}$.

For $\Delta_1 > 0$, we say an $\mathcal{R}$-admissible sequence $(k(n))_{n=1}^{n_*}$ for an interval I is *$(\mathcal{R}, \Delta_1)$-admissible* if $|r_{k(n)} \circ \cdots \circ r_{k(1)}(I)| \leqslant \Delta_1$, where $|J|$ is the length of an interval J. We call a sequence $(I_i = [x_i, x'_i] \times y_i \times \sigma_i)_{i\geqslant 1}$ of intervals in $[-2, 2]^2 \times \Sigma$ *a Δ_1-family* if there exists a family $\{(k_i(n))_{n=1}^{n_i}\}_{i\geqslant 1}$ of sequences such that $(k_i(n))_{n=1}^{n_i}$ is a fine $(\mathcal{R}, \Delta_1)$-admissible sequence for I_i for any $i \geqslant 1$, n_i tends to infinity as $i \to \infty$, and $\limsup |r_{k_i(n_i)} \circ \cdots \circ r_{k_i(1)}(I_i)| > 0$.

Let $\mathcal{F}^s(z)$ denote the leaf of $\mathcal{F}^s$ containing $z \in M$. The following is a variant of 'Denjoy property' that is shown by Pujals and Sambarino in [11].

Lemma 2.1. *There exists a constant $\Delta_1 > 0$ such that any Δ_1-family $\{I_i\}_{i=1}^{\infty}$ of intervals admits a sequence $\{z_i \in \psi(I_i)\}_{i=1}^{\infty}$ accumulating to a point of $\Omega_*^s(\Phi)$ or a periodic point z_* with $\lambda^u(z_*) > 1$.*

Proof. Almost all part of the proof of Proposition 3.1 of [11] (or Proposition 4.2 of [1]) works even if non-hyperbolic periodic orbits exist. It allow us to take a constant $\Delta_1 > 0$ such that if an interval $I = [x, x'] \times y \times \sigma$ admits an $(\mathcal{R}^{-1}, \Delta_1)$-admissible sequence $(k(n))_{n=1}^{\infty}$ then $\psi(I) \subset W^u(\Omega_*^s(\Phi))$ or $\operatorname{Int}\psi(I) \cap W^u(\mathcal{O}(z_*)) \neq \emptyset$ for some periodic point z_* with $\lambda^u(z_*) > 1$.

Let $(I_i)_{i \geqslant 1}$ be a Δ_1-family of intervals and $\{(k_i(n))_{n=1}^{n_i}\}_{i \geqslant 1}$ the corresponding family of fine $(\mathcal{R}, \Delta_1)$-admissible sequences. Put $J_i = r_{k_i(n_i)} \circ \cdots \circ r_{k_i(1)}(I_i)$ and $k_i'(n) = k_i(n_i - n + 1)$ for any $n = 1, \ldots, n_i$. Then, $(k_i'(n))_{n=1}^{n_i}$ is a fine $(\mathcal{R}^{-1}, \Delta_1)$-admissible sequence for J_i. By taking subsequences if it is necessary, we can assume that J_i converges to an interval $J_* = [\bar{x}, \bar{x}'] \times \bar{y} \times \bar{\sigma}$ and there exist sequences $(k'(n))_{n=1}^{\infty}$ and $(i_n)_{n \geqslant 1}$ such that i_n tends to infinity as $n \to \infty$ and $k_i'(n) = k'(n)$ for any $n \geqslant 1$ and $i \geqslant i_n$. It is easy to check that $(k'(n))_{n=1}^{\infty}$ is an $(\mathcal{R}^{-1}, \Delta_1)$-admissible sequence for J_*

By the choice of the constant Δ_1, there exists $x_* \in (\bar{x}, \bar{x}')$ such that $\psi(x_*, \bar{y}, \bar{\sigma}) \in W^u(\Omega_*^s(\Phi))$ or $\psi(x_*, \bar{y}, \bar{\sigma}) \in W^u(\mathcal{O}(z_*))$ for some periodic point z_* with $\lambda^u(z_*) > 1$. Hence, we can take a neighborhood U of $\bar{y}$ such that $\overline{\bigcup_{t > T} \Phi^{-t}(x_* \times U \times \bar{\sigma})}$ converges to a connected component of $\Omega_*^s(\Phi)$ or $\mathcal{O}(z_*)$ as $T \to \infty$. The lemma follows immediately. $\square$

The following is the keystone of the proof of the main theorem.

Proposition 2.2. *Let z_* be an attracting periodic point in a closed leaf L of $\mathcal{F}^s$. The leaf $\mathcal{F}^s(z)$ is a subset of $W^s(\mathcal{O}(z_*))$ for any $z \in W^s(\mathcal{O}(z_*)) \setminus L$.*

Proof. Take $z \in W^s(\mathcal{O}(z_*)) \setminus L$. Let U be the connected component of $\mathcal{F}^s(z) \cap W^s(\mathcal{O}(z_*))$ that contains z. Suppose that U does not coincide with $\mathcal{F}^s(z)$.

We say a point $z' \in M$ is accessible from U when there exists a continuous map $l : [0, 1] \to M$ such that $l(1) = z'$ and $l(t) \in U$ for any $t \in [0, 1)$. We claim any point $z_1 \in \partial U$ accessible from U is a periodic point. Without loss of generality, we may assume that $z_1 = \psi(x_0, y_0, \sigma_0)$ and $\psi((x_0, x_0') \times y_0 \times \sigma_0) \subset W^s(\mathcal{O}(z_*))$ for some $(x_0, y_0, \sigma_0) \in (-1, 1)^2 \times \Sigma$ and $x_0' \in (x_0, 1)$. Let $\Delta_1 > 0$ be the constant obtained in Lemma 2.1 and take a fine $\mathcal{R}$-admissible sequence $(k(n))_{n=1}^{\infty}$ of (x_0, y_0, σ_0). Put $C_1 = \sup\{\|Dr_k\| \mid k = 1, \ldots, k_*\}$ and $I_x = [x_0, x] \times y_0 \times \sigma_0$ for $x \in (x_0, x_0')$. Since $z_1 = \psi(x_0, y_0, \sigma_0)$ is not contained in $W^s(\mathcal{O}(z_*)))$ and $\psi((x_0, x_0') \times y_0 \times \sigma_0) \subset W^s(\mathcal{O}(z_*))$, there exist $\varepsilon_* > 0$ and an integer valued function m on (x_0, x_0') such that $(k(n))_{n=1}^{m(x)}$ is an $(\mathcal{R}, \Delta_1)$-admissible sequence for I_x and $|r_{k(m(x))} \circ \cdots \circ r_{k(1)}(I_x)| \geqslant \varepsilon_*$ for each $x \in (x_0, x_0')$. It is easy to see that $C_1^{m(x)}(x - x_0) \geqslant \varepsilon_*$, and hence, $m(x)$ tends to infinity as $x \to x_0$. It implies that there exists a Δ_1-family $\{I_i\}_{i=1}^{\infty}$ such that $I_{i+1} \subset I_i$ and $\bigcap_{i \geqslant 1} I_i = \{(x_0, y_0, \sigma_0)\}$. By the choice of Δ_1, $z_1 = \psi(x_0, y_0, \sigma_0)$ is a point of $\Omega_*^s(\Phi)$ or a periodic point with $\lambda^u(z_1) > 1$. However, in the former case, the leaf $\mathcal{F}^s(z_1)$ cannot intersect with $W^s(\mathcal{O}(z_*))$. Therefore, z_1 is a periodic point of Φ.

By the Poincaré–Bendixon theorem, $\mathcal{O}(z_1)$ is not null-homotopic in $\mathcal{F}^s(z)$. Hence, we can take a simple closed curve C in U which is not null-homotopic in $\mathcal{F}^s(z)$. On the other hand, we can see that the restriction of $\mathcal{F}^s$ on $W^s(\mathcal{O}(z_*))$ has the unique non-contractible leaf and it contains z_*. It contradicts that $C \subset U \subset W^s(\mathcal{O}(z_*)) \setminus L$. $\square$

2.2. Non-normally repelling invariant tori

In this subsection, we show the following proposition.

Proposition 2.3. *If $\mathcal{F}^s$ has a closed leaf L_0 which is not normally repelling, then it must be the Reeb foliation on $S^2 \times S^1$.*

We prepare three lemmas to prove the proposition.

Lemma 2.4. *Any periodic point z in a closed leaf L of $\mathcal{F}^s$ satisfies $\lambda^u(z) \neq 1$.*

Proof. If L is included in $\Omega_*^s(\Phi)$, then we have $\lambda^u(z) > \lambda^s(z) = 1$ for any periodic point $z \in L$. Suppose that L contains a periodic orbit and is not included in Ω_*^s. Fix a non-periodic point z' in L. By the Poincaré–Bendixon theorem, $\Phi^{-t}(z')$ converges to a periodic orbit $\mathcal{O}(z_0)$ as $t \to \infty$. It is easy to see that $\lambda^u(z_0) > \lambda^s(z_0) \geqslant 1$.

Since all periodic orbits in L are mutually isotopic when we forget their orientation, any periodic point z in L satisfies $\lambda^u(z) \neq 1$. □

Lemma 2.5. *Any closed leaf L of $\mathcal{F}^s$ is either normally repelling or contains an attracting periodic point.*

Proof. By Denjoy's theorem, if L contains no periodic points, then it is included in $\Omega_*^s(\Phi)$, and hence, it is normally repelling.

Suppose that L contains periodic points but none of them are attracting. Fix $z \in L$. Since $\Phi^t(z)$ converges to a periodic orbit $\mathcal{O}(z_0)$ as $t \to \infty$, Lemma 2.4 implies $\lambda^u(z_0) > 1$. Hence, $\|D\Phi^t|_{E^u/T\Phi(z)}\|$ goes to infinity as $t \to \infty$. By the compactness of L, it implies that L is normally repelling. □

Lemma 2.6. *Suppose that there exists a closed leaf of $\mathcal{F}^s$ with an attracting periodic point z_*. Then, $M \setminus L$ is a subset of $W^s(\mathcal{O}(z_*))$.*

Proof. Take a neighborhood U of L such that $\mathcal{F}^s(z) \cap W^s(\mathcal{O}(z_*)) \neq \emptyset$ for any $z \in U$. By Proposition 2.2, we obtain $U \setminus L \subset W^s(\mathcal{O}(z_*))$. Put $A = L \setminus W^s(\mathcal{O}(z_*))$. Since $L \cap W^s(\mathcal{O}(z_*))$ is homeomorphic to an open annulus, A is an embedded closed annulus or is an embedded circle.

Take neighborhoods V of A and V_* of $\mathcal{O}(z_*)$ which are subsets of U and are diffeomorphic to the solid torus. Since $\partial V \subset W^s(\mathcal{O}(z_*))$, there exists $T > 0$ such that $\Phi^T(\partial V) \subset \operatorname{Int} V_*$. Since $\Phi^T(\partial V)$ separates $\mathcal{O}(z_*)$ and A, it implies $\partial V_* \subset \operatorname{Int} \Phi^T(V)$. Therefore, $V_* \cup \Phi^T(V)$ is an open and closed subset of M. In particular, we obtain $M = V_* \cup \Phi^T(V)$. It implies $M \setminus L \subset V_* \cup \Phi^T(V \setminus A) \subset W^s(\mathcal{O}(z_*))$. □

Now, we prove Proposition 2.3. Suppose that $\mathcal{F}^s$ has a closed leaf L which is not normally repelling. By Lemma 2.5, the leaf L contains an attracting periodic point z_*.

By Lemma 2.6, we have $M \setminus L \subset W^s(\mathcal{O}(z_*))$. Fix a connected component B of $M \setminus L$. We claim that $\bar{B}$ is a Reeb component of $\mathcal{F}^s$. Since $W^s(\mathcal{O}(z_*)) = \bigcup_{t>0} \Phi^{-t}(U_1)$ for any sufficiently small neighborhood U_1 of $\mathcal{O}(z_*)$, we can see that B is homeomorphic to the open solid torus and it contains no closed leaf. The former implies that the torus $\partial B = L$ is compressible in $\overline{B}$. By Novikov's theorem, $\overline{B}$ must contain a Reeb component R. Since B contains no closed leaf, we obtain $\overline{B} = R$.

By the claim, we have two Reeb components R_1 and R_2 with $M = R_1 \cup R_2$. Lemma 2.4 implies that $\partial R_1 = \partial R_2$ has non-trivial linear holonomy. It implies that M must be $S^2 \times S^1$.

2.3. Proof of Theorem 1.1

Suppose that $\mathcal{F}^s$ has a Reeb component R. By Proposition 2.3, it is sufficient to show that ∂R is not normally repelling.

Suppose ∂R is normally repelling. Take an open neighborhood U of ∂R in R such that $\Phi^{-t}(\overline{U}) \subset U$ for any $t > 0$ and $\bigcap_{t>0} \Phi^{-t}(U) = \partial R$. Then, we can see that $\mathcal{F}^s(z) \setminus \bigcup_{t>0} \Phi^t(U)$ is a non-empty compact Φ-invariant subset of $\mathcal{F}^s(z)$ for each $z \in \operatorname{Int} R$. It contradicts the Poincaré–Bendixon theorem since $\mathcal{F}^s(z)$ is diffeomorphic to $\mathbb{R}^2$.

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